

CLASSIFICATION OF IDEAL SUBMANIFOLDS OF REAL SPACE FORMS WITH TYPE NUMBER ≤ 2

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ABSTRACT. Roughly speaking, an ideal immersion of a Riemannian manifold into a real space form is an isometric immersion which produces the least possible amount of tension from the ambient space at each point of the submanifold. The main purpose of this paper is to completely classify all non-minimal ideal submanifolds of real space forms with type number ≤ 2 .

1. INTRODUCTION.

Riemannian invariants play the most fundamental role in Riemannian geometry. They provide the intrinsic characteristics of Riemannian manifolds; moreover, they affect the behavior of Riemannian manifolds in general. Classically, among Riemannian curvature invariants people have studied sectional, Ricci and scalar curvatures intensively since B. Riemann.

Let $R^m(c)$ denote an m -dimensional real space form of constant sectional curvature c . Given integers $n \geq 3$ and $k \geq 1$, let $\mathcal{S}(n, k)$ be the set consisting of all unordered k -tuples (n_1, \dots, n_k) of integers ≥ 2 such that $n_1 < n$ and $n_1 + \dots + n_k \leq n$. For each $(n_1, \dots, n_k) \in \mathcal{S}(n, k)$, B.-Y. Chen introduced in [4, 5] a new type of curvature invariants, denoted by $\delta(n_1, \dots, n_k)$ (see [7] for details).

In [4, 5], Chen proved that for every n -dimensional submanifold M^n of $R^m(c)$ the invariant $\delta(n_1, \dots, n_k)$ and the squared mean curvature H^2 of M^n satisfy the following optimal fundamental inequality:

$$(1.1) \quad \delta(n_1, \dots, n_k) \leq c(n_1, \dots, n_k)H^2 + b(n_1, \dots, n_k)c,$$

where $c(n_1, \dots, n_k)$ and $b(n_1, \dots, n_k)$ are positive constants defined by

$$(1.2) \quad c(n_1, \dots, n_k) = \frac{n^2(n+k-1-\sum n_j)}{2(n+k-\sum n_j)},$$

$$(1.3) \quad b(n_1, \dots, n_k) = \frac{n(n-1)}{2} - \sum_{j=1}^k \frac{n_j(n_j-1)}{2}.$$

Many applications of the invariants $\delta(n_1, \dots, n_k)$ and of the inequality (1.1) have been obtained during the last two decades (cf. [5, 6, 7, 8] for details). For instance, it was shown that the δ -invariants give rise to new obstructions to minimal, Lagrangian and slant immersions. It was also shown that these invariants relate

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closely with the first nonzero eigenvalue λ_1 of the Laplacian Δ on M^n . Moreover, they provide an optimal lower bound of λ_1 for compact irreducible homogeneous spaces which improves a well-known result of T. Nagano [16]. Furthermore, δ -invariants have been applied in [14] by S. Haesen and L. Verstraelen to the theory of general relativity.

An isometric immersion of a Riemannian n -manifold M^n into a real space form $R^m(c)$ is called *ideal* if it satisfies the equality case of inequality (1.1) identically for some k -tuple (n_1, \dots, n_k) . Roughly speaking, an ideal immersion of a Riemannian manifold into a real space form is an isometric immersion which produces the least possible amount of tension from the ambient space at each point of the submanifold (see [5] or [7, page 269]). Since (1.1) is a very general and sharp inequality, it is a very natural and interesting problem to investigate submanifolds which verify the equality case of this inequality, i.e., to determine ideal immersions.

Recall that a hypersurface M^n of a real space form $R^{n+1}(c)$ is said to have type number $\leq r$ if the shape operator at each point $p \in M^n$ has at most r nonzero eigenvalues. In general, a submanifold M^n of $R^m(c)$ is said to have *type number* $\leq r$ if, at each point $p \in M^n$ and for each unit normal vector $\xi \in T_p^\perp M^n$, the shape operator A_ξ has at most r nonzero eigenvalues.

Since the invention of δ -invariants in early 1990s, δ -invariants and the inequalities related to these invariants have become a vibrant research subject in differential geometry. Many interesting results in this respect were obtained by many geometers (see, for instance, [1, 2, 3, 4, 5, 6, 7, 9, 11, 12, 13, 15, 17, 18]). On the other hand, the classification of ideal submanifolds in space forms remains a very challenging problem. The purpose of this paper is thus to classify ideal submanifolds of real space forms with type number ≤ 2 .

2. PRELIMINARIES.

Since $R^m(c)$ is of constant sectional curvature c , the Riemann curvature tensor \tilde{R} of $R^m(c)$ satisfies

$$\tilde{R}(X, Y)Z = c\{\langle Y, Z \rangle X - \langle X, Z \rangle Y\}.$$

Assume that $\phi : M^n \rightarrow R^m(c)$ is an isometric immersion of an n -dimensional Riemannian manifold M^n into $R^m(c)$. Denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections on M^n and $R^m(c)$, respectively.

Let X and Y be vector fields tangent to M^n and let ζ be a normal vector field of M^n . Then the formulas of Gauss and Weingarten give the following decomposition of the vector fields $\tilde{\nabla}_X Y$ and $\tilde{\nabla}_X \zeta$ into a tangent and a normal component:

$$(2.1) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$(2.2) \quad \tilde{\nabla}_X \zeta = -A_\zeta X + D_X \zeta.$$

These formulas define h , A and D which are called the second fundamental form, the shape operator and the normal connection, respectively. For each ξ , A_ξ is a symmetric endomorphism.

The shape operator and the second fundamental form are related by

$$(2.3) \quad \langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

The mean curvature vector field is defined by $H = \frac{1}{n} \text{trace } h$.

The equations of Gauss, Codazzi and Ricci are given respectively by

$$(2.4) \quad \begin{aligned} \langle R(X, Y)Z, W \rangle &= \langle A_{h(Y, Z)}X, W \rangle - \langle A_{h(X, Z)}Y, W \rangle \\ &\quad + c(\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle), \end{aligned}$$

$$(2.5) \quad (\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z),$$

$$(2.6) \quad \langle R^\perp(X, Y)\xi, \eta \rangle = \langle [A_\xi, A_\eta]X, Y \rangle$$

for X, Y, Z, W tangent to M^n and ξ, η normal to M^n , where R is the curvature tensor of M^n and $\bar{\nabla}h$ is defined by

$$(2.7) \quad (\bar{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

A submanifold M^n is said to be *totally geodesic* if $h = 0$ holds identically. It is called *totally umbilical* if its second fundamental form satisfies

$$(2.8) \quad h(X, Y) = \langle X, Y \rangle H.$$

A totally umbilical submanifold is called an extrinsic sphere if its mean curvature vector field is a parallel normal vector field, i.e., $DH = 0$ holds identically.

3. δ -INVARIANTS, INEQUALITIES AND IDEAL IMMERSIONS

Let M^n be a Riemannian n -manifold. For a plane section $\pi \subset T_p M^n$, $p \in M^n$, let $K(\pi)$ be the sectional curvature of M^n associated with π . For an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_p M^n$, the scalar curvature τ at p is defined by

$$(3.1) \quad \tau(p) = \sum_{i < j} K(e_i \wedge e_j).$$

Let L be a subspace of $T_p M^n$ with dimension $r \geq 2$ and let $\{e_1, \dots, e_r\}$ be an orthonormal basis of L . The scalar curvature $\tau(L)$ of L is defined by

$$(3.2) \quad \tau(L) = \sum_{\alpha < \beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r.$$

As before, for given integers $n \geq 3$ and $k \geq 1$, we denote by $\mathcal{S}(n, k)$ the finite set consisting of all k -tuples (n_1, \dots, n_k) of integers satisfying

$$2 \leq n_1, \dots, n_k < n \quad \text{and} \quad n_1 + \dots + n_k \leq n.$$

Moreover, we denote by $\mathcal{S}(n)$ the union $\cup_{k \geq 1} \mathcal{S}(n, k)$.

For each $(n_1, \dots, n_k) \in \mathcal{S}(n)$ and each $p \in M^n$, the invariant $\delta(n_1, \dots, n_k)(p)$ is defined by

$$(3.3) \quad \delta(n_1, \dots, n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \dots + \tau(L_k)\},$$

where L_1, \dots, L_k run over all k mutually orthogonal subspaces of $T_p M^n$ such that $\dim L_j = n_j$, $j = 1, \dots, k$.

Chen proved in [4, 5] the following sharp general relation between $\delta(n_1, \dots, n_k)$ and the squared mean curvature H^2 for submanifolds in real space forms.

Theorem A. *Let M^n be an n -dimensional submanifold of a real space form $R^m(c)$ of constant sectional curvature c . Then, for each k -tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$, we have*

$$(3.4) \quad \delta(n_1, \dots, n_k) \leq \frac{n^2(n+k-1-\sum n_j)}{2(n+k-\sum n_j)} H^2 + \frac{1}{2} \left(n(n-1) - \sum_{j=1}^k n_j(n_j-1) \right) c.$$

The equality case of inequality (3.4) holds at a point $p \in M^n$ if and only if there exists an orthonormal basis $\{e_1, \dots, e_m\}$ at p such that the shape operators of M^n in $R^m(c)$ at p with respect to $\{e_1, \dots, e_m\}$ take the following form:

$$(3.5) \quad A_r = \begin{bmatrix} A_1^r & \dots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \dots & A_k^r & \\ & & 0 & \mu_r I \end{bmatrix}, \quad r = n+1, \dots, m,$$

where I is an identity matrix and A_j^r is a symmetric $n_j \times n_j$ submatrix satisfying

$$(3.6) \quad \text{trace}(A_1^r) = \dots = \text{trace}(A_k^r) = \mu_r.$$

An isometric immersion of a Riemannian n -manifold M^n into a real space form $R^m(c)$ is called $\delta(n_1, \dots, n_k)$ -ideal if it satisfies the equality case of inequality (3.4) identically. An isometric immersion of M^n into $R^m(c)$ is called *ideal* if it is a $\delta(n_1, \dots, n_k)$ -ideal for some $(n_1, \dots, n_k) \in \mathcal{S}(n)$.

4. IDEAL SUBMANIFOLDS WITH TYPE NUMBER ≤ 2 IN \mathbb{E}^m .

In this section, we classify ideal submanifolds of \mathbb{E}^m with type number ≤ 2 .

Theorem 4.1. *Let M^n be an ideal submanifold of \mathbb{E}^m . If the type number of M^n in \mathbb{E}^m is ≤ 2 , then either M^n is a minimal submanifold or the immersion of M^n in \mathbb{E}^m is congruent to*

$$(4.1) \quad \left(\sqrt{1-a^2}x_1, x_2, \dots, x_{n-2}, ax_1 \sin x_{n-1}, ax_1 \cos x_{n-1} \sin x_n, \right. \\ \left. ax_1 \cos x_{n-1} \cos x_n, 0, \dots, 0 \right)$$

for some real number a satisfying $0 < a < 1$.

Proof. First, let us assume that M^n is a non-minimal, ideal hypersurface in \mathbb{E}^{n+1} . If M^n has type number ≤ 2 , then it follows from (3.5)-(3.6) in Theorem A that M^n is $(n_1, n - n_1)$ -ideal for some integer $n_1 \in [2, n-1]$. Thus, the shape operator $A_{e_{n+1}}$ of M^n with respect to a unit normal vector e_{n+1} has exactly one nonzero eigenvalue, say λ , with multiplicity two. Therefore, there exists a local orthonormal

frame $\{e_1, \dots, e_n\}$ of the tangent bundle of M^n such that the second fundamental form satisfies

$$(4.2) \quad h(e_{n-1}, e_{n-1}) = h(e_n, e_n) = \lambda e_{n+1}, \quad h(e_i, e_j) = 0, \text{ otherwise.}$$

Let us put $\nabla_X e_i = \sum_{j=1}^n \omega_i^j(X)$, $i = 1, \dots, n$. Then, after applying Codazzi's equation, we find

$$(4.3) \quad e_\alpha \lambda = \lambda \omega_{n-1}^\alpha(e_{n-1}) = \lambda \omega_n^\alpha(e_n), \quad e_{n-1} \lambda = e_n \lambda = 0,$$

$$(4.4) \quad \omega_{n-1}^\alpha(e_n) = \omega_n^\alpha(e_{n-1}) = 0, \quad \omega_\alpha^{n-1}(e_\beta) = \omega_\alpha^n(e_\beta) = 0,$$

for $\alpha, \beta = 1, \dots, n-2$.

Let us define distributions \mathcal{D}_1 and \mathcal{D}_2 by

$$(4.5) \quad \mathcal{D}_1 = \text{Span}\{e_1, \dots, e_{n-2}\}, \quad \mathcal{D}_2 = \text{Span}\{e_{n-1}, e_n\}.$$

It follows from (4.3) and (4.4) that \mathcal{D}_1 and \mathcal{D}_2 are integrable distributions such that leaves of \mathcal{D}_1 are totally geodesic and leaves of \mathcal{D}_2 are totally umbilical in M^n . Furthermore, (4.2) implies that leaves of \mathcal{D}_2 are also totally umbilical in \mathbb{E}^{n+1} . Hence, leaves of \mathcal{D}_2 are extrinsic spheres in \mathbb{E}^{n+1} , i.e., totally umbilical submanifolds with parallel mean curvature vector field. Now, it is easy to verify that each leaf of \mathcal{D}_2 is an extrinsic sphere in M^n . Consequently, \mathcal{D}_2 is a spherical distribution. So, by Hiepko's theorem (cf. [7, page 90]), M^n is locally a warped product $L_1 \times_f L_2$, where L_1 is a leaf of \mathcal{D}_1 , L_2 is a leaf of \mathcal{D}_2 and f is the warping function. Since L_1 is totally geodesic in M^n as well as in \mathbb{E}^{n+1} by (4.2), L_1 is an open portion of \mathbb{E}^{n-2} . Similarly, since L_2 is an extrinsic sphere of \mathbb{E}^{n+1} , L_2 is an open part of a 2-sphere. Thus, without loss of generality, we may assume that the warped product metric of $L_1 \times_f L_2$ takes the following form:

$$(4.6) \quad g = \sum_{i=1}^{n-2} dx_i^2 + f^2(x_1, \dots, x_{n-2})(dx_{n-1}^2 + \cos^2 x_{n-1} dx_n^2).$$

It is easy to see that $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ are parallel to e_1, \dots, e_n , respectively.

From (4.6) we know that the Levi-Civita connection of g satisfies

$$(4.7) \quad \begin{aligned} \nabla_{\partial_i} \partial_j &= 0, \quad i, j = 1, \dots, n-2, \\ \nabla_{\partial_i} \partial_{n-1} &= \frac{f_i}{f} \partial_{n-1}, \quad \nabla_{\partial_i} \partial_n = \frac{f_i}{f} \partial_n, \\ \nabla_{\partial_{n-1}} \partial_{n-1} &= -f \sum_{i=1}^{n-2} f_i \partial_i, \quad \nabla_{\partial_{n-1}} \partial_n = -\tan x_{n-1} \partial_n, \\ \nabla_{\partial_n} \partial_n &= -f \cos^2 x_{n-1} \sum_{i=1}^{n-2} f_i \partial_i + \frac{\sin 2x_{n-1}}{2} \partial_{n-1}, \end{aligned}$$

where $\partial_a = \frac{\partial}{\partial x_a}$, $a = 1, \dots, n$ and $f_i = \frac{\partial f}{\partial x_i}$, $i = 1, \dots, n-2$.

Gauss' equation and (4.2) imply that $\langle R(\partial_j, \partial_{n-1}) \partial_{n-1}, \partial_j \rangle = 0$, $j = 1, \dots, n-2$. On the other hand, it follows from (4.7) that $\langle R(\partial_j, \partial_{n-1}) \partial_{n-1}, \partial_j \rangle = -f f_{jj}$. Thus, we find $f_{jj} = 0$ for $j = 1, \dots, n-2$.

Similarly, we derive from (4.2), (4.7), Gauss' equation and $\langle R(\partial_i, \partial_n)\partial_n, \partial_j \rangle = 0$ ($1 \leq i \neq j \leq n-2$) that $f_{ij} = 0$. So, we have $f_{ij} = 0$ for $i, j = 1, \dots, n$. Therefore, we obtain $f = \sum_{i=1}^{n-2} b_i x_i + c$ for some real numbers b_1, \dots, b_{n-2}, c . Consequently, after applying a suitable rotation and translation, we have $f = ax_1$ for some positive number a . Thus, (4.6) and (4.7) become

$$(4.8) \quad g = \sum_{i=1}^{n-2} dx_i^2 + a^2 x_1^2 (dx_{n-1}^2 + \cos^2 x_{n-1} dx_n^2),$$

and

$$(4.9) \quad \begin{aligned} \nabla_{\partial_i} \partial_j &= 0, \quad i, j = 1, \dots, n-2, \\ \nabla_{\partial_1} \partial_{n-1} &= \frac{\partial_{n-1}}{x_1}, \quad \nabla_{\partial_1} \partial_n = \frac{\partial_n}{x_1}, \\ \nabla_{\partial_k} \partial_{n-1} &= \nabla_{\partial_k} \partial_n = 0, \quad k = 2, \dots, n-2, \\ \nabla_{\partial_{n-1}} \partial_{n-1} &= -a^2 x_1 \partial_1, \quad \nabla_{\partial_{n-1}} \partial_n = -\tan x_{n-1} \partial_n, \\ \nabla_{\partial_n} \partial_n &= -a^2 x_1 \cos^2 x_{n-1} \partial_1 + \frac{\sin 2x_{n-1}}{2} \partial_{n-1}. \end{aligned}$$

From (4.2), (4.8) and Gauss' equation we obtain

$$(4.10) \quad \langle R(\partial_{n-1}, \partial_n)\partial_n, \partial_{n-1} \rangle = a^4 x_1^4 \lambda^2 \cos^2 x_{n-1}.$$

On the other hand, it follows from (4.8) and (4.9) that

$$(4.11) \quad \langle R(\partial_{n-1}, \partial_n)\partial_n, \partial_{n-1} \rangle = a^2 (1 - a^2) x_1^2 \cos^2 x_{n-1}.$$

By combining (4.10) and (4.11) we find $\lambda^2 = (1 - a^2)/a^2 x_1^2$ with $0 < a < 1$. Thus, without loss of generality, we may put

$$(4.12) \quad \lambda = \frac{\sqrt{1 - a^2}}{ax_1},$$

which shows that λ is always nonzero. Hence, M^n is non-complete and it contains no minimal points. Moreover, (4.12) shows that the immersion of M^n in \mathbb{E}^{n+1} is rigid, since its second fundamental form is completely determined by its metric.

Let $L : M^n \rightarrow \mathbb{E}^{n+1}$ denote the immersion of M^n in \mathbb{E}^{n+1} . If we put $L_{x_\alpha} = \frac{\partial L}{\partial x_\alpha}$ and $L_{x_\alpha x_\beta} = \frac{\partial^2 L}{\partial x_\alpha \partial x_\beta}$, then (4.2), (4.8), (4.9), (4.12) and Gauss' formula yield

$$(4.13) \quad \begin{aligned} L_{x_i x_j} &= 0, \quad i, j = 1, \dots, n-2, \\ L_{x_1 x_{n-1}} &= \frac{1}{x_1} L_{x_{n-1}}, \quad L_{x_1 x_n} = \frac{1}{x_1} L_{x_n}, \\ L_{x_k x_{n-1}} &= L_{x_k x_n} = 0, \quad k = 2, \dots, n-2, \\ L_{x_{n-1} x_{n-1}} &= -a^2 x_1 L_{x_1} + a\sqrt{1 - a^2} x_1 e_{n+1}, \\ L_{x_{n-1} x_n} &= -\tan x_{n-1} L_{x_n}, \\ L_{x_n x_n} &= -a^2 x_1 \cos^2 x_{n-1} L_{x_1} + \frac{\sin 2x_{n-1}}{2} L_{x_{n-1}} \\ &\quad + a\sqrt{1 - a^2} x_1 \cos^2 x_{n-1} e_{n+1}. \end{aligned}$$

Also, from (4.2), (4.12) and Weingarten's formula, we find

$$(4.14) \quad \begin{aligned} \frac{\partial e_{n+1}}{\partial x_j} &= 0, \quad j = 1, \dots, n-2, \\ \frac{\partial e_{n+1}}{\partial x_{n-1}} &= -\frac{\sqrt{1-a^2}}{ax_1} L_{x_{n-1}}, \quad \frac{\partial e_{n+1}}{\partial x_n} = -\frac{\sqrt{1-a^2}}{ax_1} L_{x_n}. \end{aligned}$$

After solving the PDE system (4.13)-(4.14), we obtain

$$(4.15) \quad L = \sum_{\alpha=1}^{n-2} c_\alpha x_\alpha + x_1(c_{n-1} \sin x_{n-1} + c_n \cos x_{n-1} \sin x_n + c_{n+1} \cos x_{n-1} \cos x_n)$$

for some vectors $c_1, \dots, c_{n+1} \in \mathbb{E}^{n+1}$. Now, by applying (4.8) and (4.15), we may conclude that the immersion $L : M^n \rightarrow \mathbb{E}^{n+1}$ is congruent to

$$(4.16) \quad \begin{aligned} &(\sqrt{1-a^2}x_1, x_2, \dots, x_{n-2}, ax_1 \sin x_{n-1}, \\ &ax_1 \cos x_{n-1} \sin x_n, ax_1 \cos x_{n-1} \cos x_n). \end{aligned}$$

Now, let us assume that M^n is non-minimal and ideal in \mathbb{E}^m with $m \geq n+2$ and type number ≤ 2 . Then it follows from Theorem A that (4.2) holds too. Thus, it follows from (4.2) and $(\bar{\nabla}_{e_\alpha} h)(e_{n-1}, e_{n-1}) = (\bar{\nabla}_{e_{n-1}} h)(e_\alpha, e_{n-1})$ that $D_{e_\alpha} e_{n+1} = 0$ for $\alpha = 1, \dots, n-2$. Moreover, it follows from

$$\begin{aligned} (\bar{\nabla}_{e_n} h)(e_{n-1}, e_{n-1}) &= (\bar{\nabla}_{e_{n-1}} h)(e_{n-1}, e_n), \\ (\bar{\nabla}_{e_{n-1}} h)(e_n, e_n) &= (\bar{\nabla}_{e_n} h)(e_{n-1}, e_n) \end{aligned}$$

of Codazzi's equation that $D_{e_{e_{n-1}}} e_{n+1} = D_{e_n} e_{n+1} = 0$. Thus, we find $D_{e_{n+1}} = 0$, i.e., e_{n+1} is a parallel normal vector field. Because the first normal bundle is spanned by e_{n+1} , it is a parallel normal bundle. Therefore, Erbacher's reduction theorem implies that M^n is immersed in an $(n+1)$ -dimensional affine subspace of \mathbb{E}^m . Consequently, we conclude that the immersion is congruent to (4.1). \square

5. IDEAL SUBMANIFOLDS WITH TYPE NUMBER ≤ 2 IN $S^m(1)$.

Now, we classify ideal submanifolds of $S^m(1)$ with type number ≤ 2 .

Theorem 5.1. *Let M^n be an ideal submanifold of a unit m -sphere $S^m(1)$. If the type number of M^n in $S^m(1)$ is ≤ 2 , then either M^n is a minimal submanifold of $S^m(1)$ or the immersion of M^n into $S^m(1) \subset \mathbb{E}^{m+1}$ is congruent to*

$$(5.1) \quad \begin{aligned} &\left(\sqrt{1-a^2} \sin x_1, \cos x_1 \sin x_2, \dots, \sin x_{n-2} \prod_{j=1}^{n-3} \cos x_j, \prod_{j=1}^{n-2} \cos x_j, \right. \\ &\left. a \sin x_1 \sin x_{n-1}, a \sin x_1 \cos x_{n-1} \sin x_n, a \sin x_1 \cos x_{n-1} \cos x_n, 0, \dots, 0 \right) \end{aligned}$$

for some real number a satisfying $0 < a < 1$.

Proof. First, assume that M^n is a non-minimal ideal hypersurface of $S^{n+1}(1)$ with type number ≤ 2 . then as before there is a local orthonormal frame $\{e_1, \dots, e_n\}$ such that the second fundamental form of M^n in $S^m(1)$ satisfies

$$(5.2) \quad h(e_{n-1}, e_{n-1}) = h(e_n, e_n) = \lambda e_{n+1}, \quad h(e_i, e_j) = 0, \text{ otherwise.}$$

Thus, by applying the same argument as before, we know that M^n is locally a warped product $L_1 \times_f L_2$, where L_1 is an open portion of a unit $(n-2)$ -sphere and L_2 is an open portion of a 2-sphere. Hence, the warped product metric of $L_1 \times_f L_2$ takes the following form:

$$(5.3) \quad g = dx_1^2 + \cos^2 x_1 dx_2^2 + \dots + \cos^2 x_1 \dots \cos^2 x_{n-3} dx_{n-2}^2 \\ + f^2(x_1, \dots, x_{n-2})(dx_{n-1}^2 + \cos^2 x_{n-1} dx_n^2).$$

Clearly, $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ are parallel to e_1, \dots, e_n , respectively. From (5.3), we get

$$(5.4) \quad \begin{aligned} \nabla_{\partial_1} \partial_1 &= 0, \quad \nabla_{\partial_i} \partial_j = -\tan x_i \partial_j, \quad 1 \leq i < j \leq n-2, \\ \nabla_{\partial_2} \partial_2 &= \frac{\sin 2x_1}{2} \partial_1, \\ \nabla_{\partial_k} \partial_k &= \frac{\sin 2x_1}{2} \left(\prod_{j=2}^{k-1} \cos^2 x_j \right) \partial_1 + \frac{\sin 2x_2}{2} \left(\prod_{j=3}^{k-1} \cos^2 x_j \right) \partial_2 \\ &\quad + \dots + \frac{\sin 2x_{k-1}}{2} \partial_{k-1}, \quad k = 3, \dots, n-2, \\ \nabla_{\partial_i} \partial_{n-1} &= \frac{f_i}{f} \partial_{n-1}, \quad \nabla_{\partial_i} \partial_n = \frac{f_i}{f} \partial_n, \quad i = 1, \dots, n-2, \\ \nabla_{\partial_{n-1}} \partial_{n-1} &= -f \left\{ f_1 \partial_1 + f_2 \sec^2 x_1 \partial_2 + \dots + f_{n-2} \left(\prod_{i=1}^{n-3} \sec^2 x_i \right) \partial_{n-2} \right\}, \\ \nabla_{\partial_{n-1}} \partial_n &= -\tan x_{n-1} \partial_n, \\ \nabla_{\partial_n} \partial_n &= -f \cos^2 x_{n-1} \left\{ f_1 \partial_1 + \dots + f_{n-2} \left(\prod_{i=1}^{n-3} \sec^2 x_i \right) \partial_{n-2} \right\}. \end{aligned}$$

From $\langle R(\partial_j, \partial_{n-1}) \partial_{n-1}, \partial_j \rangle = \langle R(\partial_i, \partial_n) \partial_n, \partial_j \rangle = 0$, $1 \leq i \neq j \leq n-2$, (5.2) and (5.4), we obtain

$$(5.5) \quad \begin{aligned} f_{11} &= -f, \quad f_{ij} = -\tan x_i f_j, \quad 1 < i < j \leq n-2, \\ f_{22} &= \frac{\sin 2x_1}{2} f_1 - f \cos^2 x_1, \\ f_{kk} &= \frac{\sin 2x_1}{2} \cos^2 x_2 \dots \cos^2 x_{k-1} f_1 + \dots + \frac{\sin 2x_{k-2}}{2} \cos^2 x_{k-1} f_{k-2} \\ &\quad + \frac{\sin 2x_{k-1}}{2} f_{k-1} - f \prod_{j=1}^{k-1} \cos^2 x_j, \quad k = 3, \dots, n-2. \end{aligned}$$

After solving system (5.5), we get

$$(5.6) \quad \begin{aligned} f(x_1, \dots, x_{n-2}) &= b_1 \sin x_1 + b_2 \sin x_2 \cos x_1 + \dots \\ &+ b_{n-2} \sin x_{n-2} \prod_{j=1}^{n-3} \cos x_j + b_{n-1} \prod_{j=1}^{n-2} \cos x_{n-2} \end{aligned}$$

for some real numbers b_1, \dots, b_{n-1} . The function f in (5.6) is the height function of $S^{n-2}(1) \subset \mathbb{E}^{n-1}$ in the direction $v = (b_1, \dots, b_{n-1})$. So, after applying a suitable rotation of $S^{n-2}(1)$, we get $f = a \sin x_1$ for some positive number a . Thus, taking into account of (5.3) and (5.5), we have

$$(5.7) \quad \begin{aligned} g &= dx_1^2 + \cos^2 x_1 dx_2^2 + \dots + \cos^2 x_1 \dots \cos^2 x_{n-3} dx_{n-2}^2 \\ &+ a^2 \sin^2 x_1 (dx_{n-1}^2 + \cos^2 x_{n-1} dx_n^2) \end{aligned}$$

and

$$(5.8) \quad \begin{aligned} \nabla_{\partial_1} \partial_1 &= 0, \quad \nabla_{\partial_i} \partial_j = -\tan x_i \partial_j, \quad 1 \leq i < j \leq n-2, \\ \nabla_{\partial_2} \partial_2 &= \sin x_1 \cos x_1 \partial_1, \\ \nabla_{\partial_k} \partial_k &= \frac{\sin 2x_1}{2} \cos^2 x_2 \dots \cos^2 x_{k-1} \partial_1 + \frac{\sin 2x_2}{2} \cos^3 x_3 \dots \cos^2 x_{k-1} \partial_2 \\ &+ \dots + \sin x_{k-1} \cos x_{k-1} \partial_{k-1}, \quad k = 3, \dots, n-2, \\ \nabla_{\partial_1} \partial_{n-1} &= \cot x_1 \partial_{n-1}, \quad \nabla_{\partial_1} \partial_n = \cot x_1 \partial_n, \\ \nabla_{\partial_j} \partial_{n-1} &= \nabla_{\partial_j} \partial_n = 0, \quad j = 2, \dots, n-2, \\ \nabla_{\partial_{n-1}} \partial_{n-1} &= -a^2 \sin x_1 \cos x_1 \partial_1, \quad \nabla_{\partial_{n-1}} \partial_n = -\tan x_{n-1} \partial_n, \\ \nabla_{\partial_n} \partial_n &= -a^2 \sin x_1 \cos x_1 \cos^2 x_{n-1} \partial_1 + \sin x_{n-1} \cos x_{n-1} \partial_{n-1}. \end{aligned}$$

By applying Gauss' equation via (5.8), we obtain

$$(5.9) \quad \lambda = \frac{\sqrt{1-a^2}}{a} \csc x_1,$$

Let $L : M^n \rightarrow S^{n+1}(1) \subset \mathbb{E}^{n+2}$ be the immersion of M^n into \mathbb{E}^{n+2} . We obtain from (5.2), (5.7), (5.8), (5.9) and Gauss' formula that

$$(5.10) \quad \begin{aligned} L_{x_i x_j} &= -L, \quad L_{x_1 x_j} = -\tan x_1 L_j, \quad j = 2, \dots, n-2, \\ L_{x_1 x_{n-1}} &= \cot x_1 L_{x_{n-1}}, \quad L_{x_1 x_n} = \cot x_1 L_{x_n}, \\ L_{x_2 x_2} &= \sin x_1 \cos x_1 L_{x_1} - \cos^2 x_1 L, \\ L_{x_k x_k} &= \frac{\sin 2x_1}{2} \cos^2 x_2 \dots \cos^2 x_{k-1} L_{x_1} + \frac{\sin 2x_2}{2} \cos^3 x_3 \dots \cos^2 x_{k-1} L_{x_2} \\ &+ \dots + \frac{\sin 2x_{k-1}}{2} L_{x_{k-1}} - \cos^2 x_1 \dots \cos^2 x_{k-1} L, \quad k = 3, \dots, n-2, \\ L_{x_k x_{n-1}} &= L_{x_k x_n} = 0, \quad k = 2, \dots, n-2, \\ L_{x_{n-1} x_{n-1}} &= -a^2 \sin x_1 \cos x_1 L_{x_1} + a\sqrt{1-a^2} \sin x_1 e_{n+1} - a^2 \sin^2 x_1 L, \\ L_{x_{n-1} x_n} &= -\tan x_{n-1} L_{x_n}, \\ L_{x_n x_n} &= -a^2 \sin x_1 \cos x_1 \cos^2 x_{n-1} L_{x_1} + a^2 \sin x_{n-1} \cos x_{n-1} L_{x_{n-1}} \\ &+ a\sqrt{1-a^2} \sin x_1 \cos^2 x_{n-1} e_{n+1} - a^2 \sin^2 x_1 \cos^2 x_{n-1} L. \end{aligned}$$

Moreover, from (5.2), (5.7), (5.9) and Weingarten's formula we have

$$(5.11) \quad \begin{aligned} \frac{\partial e_{n+1}}{\partial x_j} &= 0, \quad j = 1, \dots, n-2, \\ \frac{\partial e_{n+1}}{\partial x_{n-1}} &= -\frac{\sqrt{1-a^2}}{a} \csc x_1 L_{x_{n-1}}, \quad \frac{\partial e_{n+1}}{\partial x_n} = -\frac{\sqrt{1-a^2}}{a} \csc x_1 L_{x_n}. \end{aligned}$$

After solving system (5.10)-(5.11) we obtain

$$(5.12) \quad \begin{aligned} L(x_1, \dots, x_n) &= c_1 \sin x_1 + c_2 \sin x_2 \cos x_1 + \dots \\ &\quad + c_{n-2} \sin x_{n-2} \prod_{j=1}^{n-3} \cos x_j + c_{n-1} \prod_{j=1}^{n-2} \cos x_j \\ &\quad + \sin x_1 (c_n \sin x_{n-1} + c_{n+1} \cos x_{n-1} \sin x_n + c_{n+2} \cos x_{n-1} \cos x_n) \end{aligned}$$

for some vectors $c_1, \dots, c_{n+1} \in \mathbb{E}^{n+2}$. Therefore, after applying (5.7) and (5.12), we conclude that L is congruent to

$$\begin{aligned} &\left(\sqrt{1-a^2} \sin x_1, \cos x_1 \sin x_2, \dots, \sin x_{n-2} \prod_{j=1}^{n-3} \cos x_j, \prod_{j=1}^{n-2} \cos x_j, \right. \\ &\quad \left. a \sin x_1 \sin x_{n-1}, a \sin x_1 \cos x_{n-1} \sin x_n, a \sin x_1 \cos x_{n-1} \cos x_n \right). \end{aligned}$$

If M^n is non-minimal and ideal in $S^m(1)$ with $m \geq n+2$ and type number ≤ 2 . Then by applying Theorem A and Codazzi's equation we know that the first normal bundle is a parallel normal bundle. Therefore, the reduction theorem implies that M^n is immersed in a totally geodesic $S^{n+1}(1) \subset S^m(1)$. Consequently, the immersion is congruent to (5.1). \square

6. IDEAL SUBMANIFOLDS WITH TYPE NUMBER ≤ 2 IN $H^m(-1)$.

Let \mathbb{E}_1^{m+1} denote the $(m+1)$ -dimensional Minkowski spacetime endowed with the Lorentzian metric

$$(6.1) \quad g = -du_1^2 + \sum_{j=2}^{m+1} du_j^2.$$

We put

$$H^m(-1) = \{u = (u_1, \dots, u_{m+1}) \in \mathbb{E}_1^{m+1} : g(u, u) = -1 \text{ and } u_1 > 0\}.$$

Then $H^m(-1)$ is the hyperbolic m -space of constant sectional curvature -1 .

In this section, we classify ideal submanifolds of $H^m(-1)$ with type number ≤ 2 .

Theorem 6.1. *Let $\phi : M^n \rightarrow H^m(-1)$ be an ideal immersion of a Riemannian n -manifold into $H^m(-1)$. If M^n has type number ≤ 2 , then either M^n is minimal in $H^m(-1)$ or $\phi(M^n)$ lies in a totally geodesic $H^{n+1}(-1) \subset H^m(-1)$. Moreover, in the later case the corresponding immersion $L : M^n \rightarrow H^{n+1}(-1) \subset \mathbb{E}_1^{n+2}$ is congruent to one of the following three immersions:*

$$\begin{aligned}
(A) \quad & \left\{ \begin{aligned} & \left(\frac{ab \sinh x_{n-2} + (1+b^2) \cosh x_{n-2}}{\sqrt{1+b^2}} \prod_{j=1}^{n-3} \cosh x_j, \sinh x_1, \sinh x_2 \cosh x_1, \right. \\ & \quad \dots, \sinh x_{n-3} \prod_{j=1}^{n-4} \cosh x_j, \frac{\sqrt{1-a^2+b^2}}{\sqrt{1+b^2}} \sinh x_{n-2} \prod_{j=1}^{n-3} \cosh x_j, \\ & \quad (a \sinh x_{n-2} + b \cosh x_{n-2}) \cos x_{n-1} \cos x_n \prod_{j=1}^{n-3} \cosh x_j, \\ & \quad (a \sinh x_{n-2} + b \cosh x_{n-2}) \cos x_{n-1} \sin x_n \prod_{j=1}^{n-3} \cosh x_j, \\ & \quad \left. (a \sinh x_{n-2} + b \cosh x_{n-2}) \sin x_{n-1} \prod_{j=1}^{n-3} \cosh x_j \right), \quad a^2 < 1+b^2; \end{aligned} \right. \\
(B) \quad & \left\{ \begin{aligned} & \left(\frac{a(b^4-4+4b^2(x_{n-1}^2+x_n^2)) \sinh x_{n-2} + b(b^4+4+4b^2(x_{n-1}^2+x_n^2)) \cosh x_{n-2}}{4b^3 \prod_{j=1}^{n-3} \operatorname{sech} x_j}, \right. \\ & \quad \frac{a(b^4+4-4b^2(x_{n-1}^2+x_n^2)) \sinh x_{n-2} + b(b^4-4-4b^2(x_{n-1}^2+x_n^2)) \cosh x_{n-2}}{4b^3 \prod_{j=1}^{n-3} \operatorname{sech} x_j}, \\ & \quad \sinh x_1, \dots, \sinh x_{n-3} \prod_{j=1}^{n-4} \cosh x_j, \frac{\sqrt{b^2-a^2}}{b} \sinh x_{n-2} \prod_{j=1}^{n-3} \cosh x_j, \\ & \quad (a \sinh x_{n-2} + b \cosh x_{n-2}) x_{n-1} \prod_{j=1}^{n-3} \cosh x_j, \\ & \quad \left. (a \sinh x_{n-2} + b \cosh x_{n-2}) x_n \prod_{j=1}^{n-3} \cosh x_j \right), \quad a^2 < b^2; \end{aligned} \right. \\
(C) \quad & \left\{ \begin{aligned} & \left((a \sinh x_{n-2} + b \cosh x_{n-2}) \cosh x_{n-1} \cosh x_n \prod_{j=1}^{n-3} \cosh x_j, \right. \\ & \quad (a \sinh x_{n-2} + b \cosh x_{n-2}) \cosh x_{n-1} \sinh x_n \prod_{j=1}^{n-3} \cosh x_j, \\ & \quad (a \sinh x_{n-2} + b \cosh x_{n-2}) \sinh x_{n-1} \prod_{j=1}^{n-3} \cosh x_j, \\ & \quad \sinh x_1, \sinh x_2 \cosh x_1, \dots, \sinh x_{n-3} \prod_{j=1}^{n-4} \cosh x_j, \\ & \quad \left. \frac{\sqrt{b^2-a^2-1}}{\sqrt{1+a^2}} \prod_{j=1}^{n-2} \cosh x_j, \frac{ab \cosh x_{n-2} + (1+a^2) \sinh x_{n-2}}{\sqrt{1+a^2}} \prod_{j=1}^{n-3} \cosh x_j \right), \\ & \quad 1+a^2 < b^2. \end{aligned} \right.
\end{aligned}$$

Proof. Assume that M^n is a non-minimal ideal hypersurface of $H^{n+1}(-1)$ with type number ≤ 2 . Then there is a local orthonormal frame $\{e_1, \dots, e_n\}$ such that

$$(6.2) \quad h(e_{n-1}, e_{n-1}) = h(e_n, e_n) = \lambda e_{n+1}, \quad h(e_i, e_j) = 0, \text{ otherwise.}$$

By applying the same argument as before, we know that M^n is locally a warped product $L_1 \times_f L_2$, where L_1 is an open portion of $H^{n-2}(-1)$ and L_2 is a totally umbilical surface of $H^{n+1}(-1)$. Therefore, the warped product metric of $L_1 \times_f L_2$ takes one of the following three forms:

$$(6.3) \quad g_1 = dx_1^2 + \cosh^2 x_1 dx_2^2 + \dots + \cosh^2 x_1 \dots \cosh^2 x_{n-3} dx_{n-2}^2 \\ + f^2(x_1, \dots, x_{n-2})(dx_{n-1}^2 + \cos^2 x_{n-1} dx_n^2);$$

$$(6.4) \quad g_2 = dx_1^2 + \cosh^2 x_1 dx_2^2 + \dots + \cosh^2 x_1 \dots \cosh^2 x_{n-3} dx_{n-2}^2 \\ + f^2(x_1, \dots, x_{n-2})(dx_{n-1}^2 + dx_n^2);$$

$$(6.5) \quad g_3 = dx_1^2 + \cosh^2 x_1 dx_2^2 + \dots + \cosh^2 x_1 \dots \cosh^2 x_{n-3} dx_{n-2}^2 \\ + f^2(x_1, \dots, x_{n-2})(dx_{n-1}^2 + \cosh^2 x_{n-1} dx_n^2).$$

Case (1): The metric of $L_1 \times_f L_2$ is (6.3). In this case, we find

$$(6.6) \quad \begin{aligned} \nabla_{\partial_1} \partial_1 &= 0, \\ \nabla_{\partial_i} \partial_j &= \tanh x_i \partial_j, \quad 1 \leq i < j \leq n-2, \\ \nabla_{\partial_2} \partial_2 &= -\sinh x_1 \cosh x_1 \partial_1, \\ \nabla_{\partial_k} \partial_k &= -\frac{\sinh 2x_1}{2} \prod_{j=2}^{k-1} \cosh^2 x_j \partial_1 - \frac{\sinh 2x_2}{2} \prod_{j=3}^{k-1} \cosh^2 x_j \partial_2 \\ &\quad - \dots - \sinh x_{k-1} \cosh x_{k-1} \partial_{k-1}, \quad k = 3, \dots, n-2, \\ \nabla_{\partial_i} \partial_{n-1} &= \frac{f_i}{f} \partial_{n-1}, \quad \nabla_{\partial_i} \partial_n = \frac{f_i}{f} \partial_n, \quad i = 1, \dots, n-2, \\ \nabla_{\partial_{n-1}} \partial_{n-1} &= -f \left\{ f_1 \partial_1 + f_2 \operatorname{sech}^2 x_1 \partial_2 + \dots + f_{n-2} \prod_{i=1}^{n-3} \operatorname{sech}^2 x_i \partial_{n-2} \right\}, \\ \nabla_{\partial_{n-1}} \partial_n &= -\tan x_{n-1} \partial_n, \\ \nabla_{\partial_n} \partial_n &= -f \cos^2 x_{n-1} \left\{ f_1 \partial_1 + \dots + f_{n-2} \prod_{i=1}^{n-3} \operatorname{sech}^2 x_i \partial_{n-2} \right\}. \end{aligned}$$

From $\langle R(\partial_j, \partial_{n-1}) \partial_{n-1}, \partial_j \rangle = \langle R(\partial_i, \partial_n) \partial_n, \partial_j \rangle = 0$, $1 \leq i \neq j \leq n-2$, (6.2) and (6.6), it follows that

$$(6.7) \quad \begin{aligned} f_{11} &= f, \quad f_{ij} = \tanh x_i f_j, \quad 1 < i < j \leq n-2, \\ f_{22} + \sinh x_1 \cosh x_1 f_1 &= f \cosh^2 x_1, \\ f_{kk} + f_1 \sinh x_1 \cosh x_1 \cosh^2 x_2 \dots \cosh^2 x_{k-1} + \dots \\ &\quad + f_{k-2} \sinh x_{k-2} \cosh x_{k-2} \cosh^2 x_{k-1} + f_{k-1} \sinh x_{k-1} \cosh x_{k-1} \\ &= f \cosh^2 x_1 \dots \cosh^2 x_{k-1}, \quad k = 3, \dots, n-2. \end{aligned}$$

After solving system (6.7), we obtain

$$(6.8) \quad \begin{aligned} f(x_1, \dots, x_{n-2}) &= b_1 \sinh x_1 + b_2 \sinh x_2 \cosh x_1 + \dots \\ &+ b_{n-2} \sinh x_{n-2} \prod_{j=1}^{n-3} \cosh x_j + b_{n-1} \prod_{j=1}^{n-2} \cosh x_j \end{aligned}$$

for some real numbers b_1, \dots, b_{n-1} . So, after applying a suitable rotation of $H^{n+1}(-1)$, we get

$$(6.9) \quad f = (a \sinh x_{n-2} + b \cosh x_{n-2}) \prod_{j=1}^{n-3} \cosh x_j$$

for some real numbers a and b , not both zero. Consequently, we obtain

$$(6.10) \quad \begin{aligned} g_1 &= dx_1^2 + \cosh^2 x_1 dx_2^2 + \dots + \cosh^2 x_1 \dots \cosh^2 x_{n-3} dx_{n-2}^2 \\ &+ (a \sinh x_{n-2} + b \cosh x_{n-2})^2 \prod_{j=1}^{n-3} \cosh^2 x_j (dx_{n-1}^2 + \cos^2 x_{n-1} dx_n^2) \end{aligned}$$

which implies that the Levi-Civita connection satisfies

$$(6.11) \quad \begin{aligned} \nabla_{\partial_1} \partial_1 &= 0, \quad \nabla_{\partial_i} \partial_j = \tanh x_i \partial_j, \quad 1 \leq i < j \leq n-2, \\ \nabla_{\partial_2} \partial_2 &= -\frac{\sinh 2x_1}{2} \partial_1, \\ \nabla_{\partial_k} \partial_k &= -\sum_{j=1}^{k-1} \frac{\sinh 2x_j}{2} \prod_{i=j+1}^{k-1} \cosh^2 x_i \partial_j, \quad k = 3, \dots, n-2, \\ \nabla_{\partial_s} \partial_r &= \tanh x_s \partial_r, \quad s = 1, \dots, n-3, \quad r = n-1, n, \\ \nabla_{\partial_{n-2}} \partial_r &= \frac{a \cosh x_{n-2} + b \sinh x_{n-2}}{a \sinh x_{n-2} + b \cosh x_{n-2}} \partial_r, \quad r = n-1, n, \\ \nabla_{\partial_{n-1}} \partial_{n-1} &= -(a \sinh x_{n-2} + b \cosh x_{n-2})^2 \left\{ \frac{\sinh 2x_1}{2} \prod_{j=2}^{n-3} \cosh^2 x_j \partial_1 \right. \\ &\quad \left. + \frac{\sinh 2x_2}{2} \prod_{j=3}^{n-3} \cosh^2 x_j \partial_2 + \dots + \frac{\sinh 2x_{n-3}}{2} \partial_{n-3} \right\} \\ &\quad - (a \sinh x_{n-2} + b \cosh x_{n-2})(a \cosh x_{n-2} + b \sinh x_{n-2}) \partial_{n-2}, \\ \nabla_{\partial_{n-1}} \partial_n &= -\tanh x_{n-1} \partial_n, \\ \nabla_{\partial_n} \partial_n &= -(a \sinh x_{n-2} + b \cosh x_{n-2})^2 \cos^2 x_{n-1} \left\{ \frac{\sinh 2x_1}{2} \prod_{j=2}^{n-3} \cosh^2 x_j \partial_1 \right. \\ &\quad \left. + \frac{\sinh 2x_2}{2} \prod_{j=3}^{n-3} \cosh^2 x_j \partial_2 + \dots + \frac{\sinh 2x_{n-3}}{2} \partial_{n-3} \right\} \\ &\quad - (a \sinh x_{n-2} + b \cosh x_{n-2})(a \cosh x_{n-2} + b \sinh x_{n-2}) \cos^2 x_{n-1} \partial_{n-2} \\ &\quad + \frac{\sin 2x_{n-1}}{2} \partial_{n-1}. \end{aligned}$$

From the equation $\langle R(\partial_{n-1}, \partial_n) \partial_n, \partial_{n-1} \rangle = (\lambda^2 - 1)g_{n-1n-1}g_{nn}$ of Gauss and (6.11) we find

$$(6.12) \quad \lambda^2 = \frac{(1 - a^2 + b^2)}{(a \sinh x_{n-2} + b \cosh x_{n-2})^2} \prod_{j=1}^{n-3} \operatorname{sech}^2 x_j,$$

which implies $a^2 < 1 + b^2$. Thus, we may put

$$(6.13) \quad \lambda = \frac{\sqrt{1 - a^2 + b^2}}{a \sinh x_{n-2} + b \cosh x_{n-2}} \prod_{j=1}^{n-3} \operatorname{sech} x_j, \quad a^2 < 1 + b^2.$$

Hence, M^n does not contain minimal points in $H^{n+1}(-1)$.

Let $L : M^n \rightarrow H^{n+1}(-1) \subset \mathbb{E}_1^{n+2}$ denote the immersion of M^n into \mathbb{E}_1^{n+2} . Then we get from (6.2), (6.10), (6.11), (6.13) and Gauss' formula that

$$\begin{aligned}
(6.14) \quad & L_{x_i x_j} = L, \quad L_{x_1 x_j} = \tanh x_1 L_j, \quad j = 2, \dots, n-2, \\
& L_{x_2 x_2} = -\sinh x_1 \cosh x_1 L_{x_1} + \cosh^2 x_1 L, \\
& L_{x_k x_k} = \prod_{j=1}^{k-1} \cosh^2 x_j L - \sum_{j=1}^{k-1} \frac{\sinh 2x_j}{2} \prod_{i=j+1}^{k-1} \cosh^2 x_i L_j, \quad k = 3, \dots, n-2, \\
& L_{x_s x_r} = \tanh x_s L_{x_r}, \quad s = 1, \dots, n-3, \quad r = n-1, n, \\
& L_{x_{n-2} x_r} = \frac{a \cosh x_{n-2} + b \sinh x_{n-2}}{a \sinh x_{n-2} + b \cosh x_{n-2}} L_{x_r}, \quad r = n-1, n, \\
& L_{x_{n-1} x_{n-1}} = (a \sinh x_{n-2} + b \cosh x_{n-2})^2 \prod_{j=1}^{n-3} \cosh^2 x_j L \\
& \quad - (a \sinh x_{n-2} + b \cosh x_{n-2})^2 \left\{ \frac{\sinh 2x_1}{2} \prod_{j=2}^{n-3} \cosh^2 x_j L_{x_1} \right. \\
& \quad \left. + \frac{\sinh 2x_2}{2} \prod_{j=3}^{n-3} \cosh^2 x_j L_{x_2} + \dots + \frac{\sinh 2x_{n-3}}{2} L_{x_{n-3}} \right\} \\
& \quad - (a \sinh x_{n-2} + b \cosh x_{n-2})(a \cosh x_{n-2} + b \sinh x_{n-2}) L_{x_{n-2}} \\
& \quad + \sqrt{1 - a^2 + b^2} (a \sinh x_{n-2} + b \cosh x_{n-2}) \prod_{j=1}^{n-3} \cosh x_j e_{n+1}, \\
& L_{x_{n-1} x_n} = -\tanh x_{n-1} L_{x_n}, \\
& L_{x_n x_n} = (a \sinh x_{n-2} + b \cosh x_{n-2})^2 \cos^2 x_{n-1} \prod_{j=1}^{n-3} \cosh^2 x_j L \\
& \quad - (a \sinh x_{n-2} + b \cosh x_{n-2})^2 \cos^2 x_{n-1} \left\{ \frac{\sinh 2x_1}{2} \prod_{j=2}^{n-3} \cosh^2 x_j L_{x_1} \right. \\
& \quad \left. + \frac{\sinh 2x_2}{2} \prod_{j=3}^{n-3} \cosh^2 x_j L_{x_2} + \dots + \frac{\sinh 2x_{n-3}}{2} L_{x_{n-3}} \right\} \\
& \quad - (a \sinh x_{n-2} + b \cosh x_{n-2})(a \cosh x_{n-2} + b \sinh x_{n-2}) \cos^2 x_{n-1} L_{x_{n-2}}
\end{aligned}$$

$$\begin{aligned}
& + \sin x_{n-1} \cos x_{n-1} L_{x_{n-1}} \\
& + \sqrt{1-a^2+b^2} (a \sinh x_{n-2} + b \cosh x_{n-2}) \cos^2 x_{n-1} \prod_{j=1}^{n-3} \cosh x_j e_{n+1}.
\end{aligned}$$

Moreover, it follows from (6.2), (6.10), (6.13) and Weingarten's formula that

$$\begin{aligned}
(6.15) \quad & \frac{\partial e_{n+1}}{\partial x_j} = 0, \quad j = 1, \dots, n-2, \\
& \frac{\partial e_{n+1}}{\partial x_r} = -\frac{\sqrt{1-a^2+b^2} \operatorname{sech} x_1 \cdots \operatorname{sech} x_{n-3}}{a \sinh x_{n-2} + b \cosh x_{n-2}} L_{x_r}, \quad r = n-1, n.
\end{aligned}$$

Solving system (6.14)-(6.15) gives

$$\begin{aligned}
(6.16) \quad & L(x_1, \dots, x_n) = c_1 \sinh x_1 + \cosh x_1 \left\{ c_2 \sinh x_2 + c_3 \sinh x_3 \cosh x_2 \right. \\
& + \cdots + c_{n-2} \sinh x_{n-2} \prod_{j=2}^{n-3} \cosh x_j + c_{n-1} \prod_{j=2}^{n-2} \cosh x_j \left. \right\} \\
& + (a \sinh x_{n-2} + b \cosh x_{n-2}) \prod_{j=1}^{n-3} \cosh x_j \left\{ c_n \sin x_{n-1} \right. \\
& \left. + c_{n+1} \cos x_{n-1} \sin x_n + c_{n+2} \cos x_{n-1} \cos x_n \right\}.
\end{aligned}$$

Now, we conclude from (6.10) and (6.16) that L is congruent to immersion (A).

If M^n is non-minimal and ideal in $H^m(-1)$ with type number ≤ 2 , then it follows from Theorem A, Codazzi's equation and Reduction Theorem that M^n is immersed in a totally geodesic $H^{n+1}(-1) \subset H^m(-1)$. Therefore, we obtain the same conclusion.

Case (2): The metric of $L_1 \times_f L_2$ is (6.4). In this case, we find in the same way as case (1) that the warping function is given by (6.8). Thus, without loss of generality, we may assume that the metric tensor is given by

$$\begin{aligned}
(6.17) \quad & g_2 = dx_1^2 + \cosh^2 x_1 dx_2^2 + \cdots + \cosh^2 x_1 \cdots \cosh^2 x_{n-3} dx_{n-2}^2 \\
& + (a \sinh x_{n-2} + b \cosh x_{n-2})^2 \prod_{j=1}^{n-3} \cosh^2 x_j (dx_{n-1}^2 + dx_n^2).
\end{aligned}$$

Now, by applying a similar argument as case (1), we get

$$(6.18) \quad \lambda = \frac{\sqrt{b^2 - a^2}}{a \sinh x_{n-2} + b \cosh x_{n-2}} \prod_{j=1}^{n-3} \operatorname{sech} x_j$$

for some real numbers a, b satisfying $b^2 > a^2$. Hence, we obtain the following system for the immersion $L : M^n \rightarrow H^{n+1}(-1) \subset \mathbb{E}_1^{n+2}$.

$$\begin{aligned}
L_{x_i x_j} &= L, \quad L_{x_1 x_j} = \tanh x_1 L_j, \quad j = 2, \dots, n-2, \\
L_{x_2 x_2} &= -\sinh x_1 \cosh x_1 L_{x_1} + \cosh^2 x_1 L, \\
L_{x_k x_k} &= \prod_{j=1}^{k-1} \cosh^2 x_j L - \sum_{j=1}^{k-1} \frac{\sinh 2x_j}{2} \prod_{i=j+1}^{k-1} \cosh^2 x_i L_j, \quad k = 3, \dots, n-2, \\
L_{x_s x_r} &= \tanh x_s L_{x_r}, \quad s = 1, \dots, n-3, \quad r = n-1, n, \\
L_{x_{n-2} x_r} &= \frac{a \cosh x_{n-2} + b \sinh x_{n-2}}{a \sinh x_{n-2} + b \cosh x_{n-2}} L_{x_r}, \quad r = n-1, n, \\
L_{x_{n-1} x_{n-1}} &= (a \sinh x_{n-2} + b \cosh x_{n-2})^2 \prod_{j=1}^{n-3} \cosh^2 x_j L \\
&\quad - (a \sinh x_{n-2} + b \cosh x_{n-2})^2 \left\{ \frac{\sinh 2x_1}{2} \prod_{j=2}^{n-3} \cosh^2 x_j L_{x_1} \right. \\
&\quad \left. + \frac{\sinh 2x_2}{2} \prod_{j=3}^{n-3} \cosh^2 x_j L_{x_2} + \dots + \frac{\sinh 2x_{n-3}}{2} L_{x_{n-3}} \right\} \\
(6.19) \quad &\quad - (a \sinh x_{n-2} + b \cosh x_{n-2})(a \cosh x_{n-2} + b \sinh x_{n-2}) L_{x_{n-2}} \\
&\quad + \sqrt{b^2 - a^2} (a \sinh x_{n-2} + b \cosh x_{n-2}) \prod_{j=1}^{n-3} \cosh x_j e_{n+1}, \\
L_{x_{n-1} x_n} &= 0, \\
L_{x_n x_n} &= (a \sinh x_{n-2} + b \cosh x_{n-2})^2 \prod_{j=1}^{n-3} \cosh^2 x_j L \\
&\quad - (a \sinh x_{n-2} + b \cosh x_{n-2})^2 \left\{ \frac{\sinh 2x_1}{2} \prod_{j=2}^{n-3} \cosh^2 x_j L_{x_1} \right. \\
&\quad \left. + \frac{\sinh 2x_2}{2} \prod_{j=3}^{n-3} \cosh^2 x_j L_{x_2} + \dots + \frac{\sinh 2x_{n-3}}{2} L_{x_{n-3}} \right\} \\
&\quad - (a \sinh x_{n-2} + b \cosh x_{n-2})(a \cosh x_{n-2} + b \sinh x_{n-2}) L_{x_{n-2}} \\
&\quad + \sqrt{b^2 - a^2} (a \sinh x_{n-2} + b \cosh x_{n-2}) \prod_{j=1}^{n-3} \cosh x_j e_{n+1},
\end{aligned}$$

and

$$\begin{aligned}
(6.20) \quad &\frac{\partial e_{n+1}}{\partial x_j} = 0, \quad j = 1, \dots, n-2, \\
&\frac{\partial e_{n+1}}{\partial x_r} = -\frac{\sqrt{b^2 - a^2} \operatorname{sech} x_1 \cdots \operatorname{sech} x_{n-3}}{a \sinh x_{n-2} + b \cosh x_{n-2}} L_{x_r}, \quad r = n-1, n.
\end{aligned}$$

After solving system (6.19)-(6.20), we obtain

$$\begin{aligned}
(6.21) \quad L(x_1, \dots, x_n) = & c_1 \sinh x_1 + \cosh x_1 \left\{ c_2 \sinh x_2 + c_3 \sinh x_3 \cosh x_2 \right. \\
& + \dots + c_{n-2} \sinh x_{n-2} \prod_{j=2}^{n-3} \cosh x_j + c_{n-1} \prod_{j=2}^{n-2} \cosh x_j \left. \right\} \\
& + (a \sinh x_{n-2} + b \cosh x_{n-2}) \left(\prod_{j=1}^{n-3} \cosh x_j \right) \times \\
& \{ c_n x_{n-1} + c_{n+1} x_n + c_{n+2} (x_{n-1}^2 + x_n^2) \}.
\end{aligned}$$

So, by using (6.17) and (6.21) we conclude that L is congruent to immersion (B).

Case (3): The metric of $L_1 \times_f L_2$ is (6.5). In this case, we find in the same way as case (1) that the warping function is given by (6.8). Thus, without loss of generality, we may assume that the metric tensor is given by

$$\begin{aligned}
(6.22) \quad g_3 = & dx_1^2 + \cosh^2 x_1 dx_2^2 + \dots + \cosh^2 x_1 \dots \cosh^2 x_{n-3} dx_{n-2}^2 \\
& + (a \sinh x_{n-2} + b \cosh x_{n-2})^2 \prod_{j=1}^{n-3} \cosh^2 x_j (dx_{n-1}^2 + \cosh^2 x_{n-1} dx_n^2).
\end{aligned}$$

Now, by applying a similar argument as case (1), we get

$$(6.23) \quad \lambda = \frac{\sqrt{b^2 - a^2 - 1}}{a \sinh x_{n-2} + b \cosh x_{n-2}} \prod_{j=1}^{n-3} \operatorname{sech} x_j, \quad b^2 > 1 + a^2.$$

Thus, we may derive the following system for $L : M^n \rightarrow H^{n+1}(-1) \subset \mathbb{E}_1^{n+2}$.

$$\begin{aligned}
(6.24) \quad & L_{x_i x_j} = L, \quad L_{x_1 x_j} = \tanh x_1 L_j, \quad j = 2, \dots, n-2, \\
& L_{x_2 x_2} = -\sinh x_1 \cosh x_1 L_{x_1} + \cosh^2 x_1 L, \\
& L_{x_k x_k} = \prod_{j=1}^{k-1} \cosh^2 x_j L - \sum_{j=1}^{k-1} \frac{\sinh 2x_j}{2} \prod_{i=j+1}^{k-1} \cosh^2 x_i L_j, \quad k = 3, \dots, n-2, \\
& L_{x_s x_r} = \tanh x_s L_{x_r}, \quad s = 1, \dots, n-3, \quad r = n-1, n, \\
& L_{x_{n-2} x_r} = \frac{a \cosh x_{n-2} + b \sinh x_{n-2}}{a \sinh x_{n-2} + b \cosh x_{n-2}} L_{x_r}, \quad r = n-1, n, \\
& L_{x_{n-1} x_{n-1}} = (a \sinh x_{n-2} + b \cosh x_{n-2})^2 \left\{ \prod_{j=1}^{n-3} \cosh^2 x_j L - \frac{\sinh 2x_1}{2} \prod_{j=2}^{n-3} \cosh^2 x_j L_{x_1} \right. \\
& \quad - \frac{\sinh 2x_2}{2} \prod_{j=3}^{n-3} \cosh^2 x_j L_{x_2} - \dots - \frac{\sinh 2x_{n-3}}{2} L_{x_{n-3}} \left. \right\} \\
& \quad - (a \sinh x_{n-2} + b \cosh x_{n-2})(a \cosh x_{n-2} + b \sinh x_{n-2}) L_{x_{n-2}} \\
& \quad + \sqrt{b^2 - a^2 - 1} (a \sinh x_{n-2} + b \cosh x_{n-2}) \prod_{j=1}^{n-3} \cosh x_j e_{n+1},
\end{aligned}$$

$$\begin{aligned}
L_{x_{n-1}x_n} &= \tanh x_{n-1} L_{x_n}, \\
L_{x_n x_n} &= (a \sinh x_{n-2} + b \cosh x_{n-2})^2 \cosh^2 x_{n-1} \left\{ \prod_{j=1}^{n-3} \cosh^2 x_j L \right. \\
&\quad \left. - \frac{\sinh 2x_1}{2} \prod_{j=2}^{n-3} \cosh^2 x_j L_{x_1} - \frac{\sinh 2x_2}{2} \prod_{j=3}^{n-3} \cosh^2 x_j L_{x_2} - \cdots - \frac{\sinh 2x_{n-3}}{2} L_{x_{n-3}} \right\} \\
&\quad - (a \sinh x_{n-2} + b \cosh x_{n-2})(a \cosh x_{n-2} + b \sinh x_{n-2}) \cosh^2 x_{n-1} L_{x_{n-2}} \\
&\quad - \sinh x_{n-1} \cosh x_{n-1} L_{x_{n-1}} \\
&\quad + \sqrt{b^2 - a^2 - 1} (a \sinh x_{n-2} + b \cosh x_{n-2}) \cosh^2 x_{n-1} \prod_{j=1}^{n-3} \cosh x_j e_{n+1},
\end{aligned}$$

and

$$\begin{aligned}
(6.25) \quad \frac{\partial e_{n+1}}{\partial x_j} &= 0, \quad j = 1, \dots, n-2, \\
\frac{\partial e_{n+1}}{\partial x_r} &= -\frac{\sqrt{b^2 - a^2 - 1} \operatorname{sech} x_1 \cdots \operatorname{sech} x_{n-3}}{a \sinh x_{n-2} + b \cosh x_{n-2}} L_{x_r}, \quad r = n-1, n.
\end{aligned}$$

After solving system (6.24)-(6.25), we get

$$\begin{aligned}
(6.26) \quad L(x_1, \dots, x_n) &= c_1 \sinh x_1 + \cosh x_1 \left\{ c_2 \sinh x_2 + c_3 \sinh x_3 \cosh x_2 \right. \\
&\quad \left. + \cdots + c_{n-2} \sinh x_{n-2} \prod_{j=2}^{n-3} \cosh x_j + c_{n-1} \prod_{j=2}^{n-2} \cosh x_j \right\} \\
&\quad + (a \sinh x_{n-2} + b \cosh x_{n-2}) \left(\prod_{j=1}^{n-3} \cosh x_j \right) \{ c_n \sinh x_{n-1} \\
&\quad + c_{n+1} \cosh x_{n-1} \sinh x_n + c_{n+2} \cosh x_{n-1} \cosh x_n \}.
\end{aligned}$$

Consequently, we conclude from (6.22) and (6.26) that L is congruent to immersion (C).

If M^n is non-minimal and ideal in $H^m(-1)$ with type number ≤ 2 , then it follows again from Theorem A, Codazzi's equation and Erbacher's reduction theorem that M^n is immersed in a totally geodesic $H^{n+1}(-1) \subset H^m(-1)$ for both cases (2) and (3) as well. Therefore, we obtain the same conclusion as above. \square

Remark 6.1. Minimal ideal hypersurfaces of $S^{n+1}(1)$ with type number ≤ 2 are either totally geodesic in $S^{n+1}(1)$ or they are given by case (2) of [10, Theorem 2]. Similarly, minimal ideal hypersurfaces of $H^{n+1}(-1)$ with type number ≤ 2 are either totally geodesic in $H^{n+1}(-1)$ or they are given by case (3) of [10, Theorem 3] (cf. [7, page 423]). Moreover, by using Codazzi's equation, it is easy to verify that each minimal ideal submanifold with type number ≤ 2 in $S^m(1)$ (respectively, in $H^m(-1)$) is contained in a totally geodesic $S^{n+1}(1) \subset S^m(1)$ (respectively, in a totally geodesic $H^{n+1}(-1) \subset H^m(-1)$).

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